ON THE SADDLE POINT PROPERTY OF ABRESCH-LANGER CURVES UNDER THE CURVE SHORTENING FLOW

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ABSTRACT. In the study of the curve shortening flow on general closed curves, Abresch and Langer posed a conjecture that the homothetic curves can be regarded as saddle points between multi-folded circles and some singular curves. In other words, these homothetic curves are the watershed between curves with a nonsingular future and those with singular future along the flow. In this article, we provide an affirmitive proof to this conjecture.

Let γ_0 be a given immersed closed plane curve. We consider the initial value problem of the curve shortening flow

$$\begin{cases} \gamma_t(p,t) = -\kappa(p,t)\boldsymbol{\nu}(p,t) \\ \\ \gamma(p,0) = \gamma_0(p) \end{cases} \tag{1}$$

where $\gamma(p,t)$ is a family of curves with curvature κ and unit (outward) normal $\boldsymbol{\nu}(p,t)$. The curve shortening flow for an embedded closed initial curve was completely characterized by the Grayson convexity theorem, [Gr1], and the Gage-Hamilton theorem, [GH]. The first asserts that the flow drives any such γ_0 to a convex curve while the second says that a convex curve shrinks to a round point. However, when γ_0 has self-intersections, it is easy to see that singularities may arise in the process. A typical example is the flow of the cardoid. The little loop of the cardoid contracts and develops a cusp when the large loop still exists. It is thus necessary to classify the singularity of the flow. A natural way of classification arises from the blow-up rate of the curvature into type I and type II singularities (Altschuler [Al] and a series of important studies of Angenent, for

examples, [An3] and [AV]). In our situation, all the singularities are of type I, namely, $|\kappa|_{\max}(t)\sqrt{t_\infty - t} \le C$, where t_∞ is the blow-up time, the singularity looks asymptotically like a contracting self-similar solution.

A (contracting) self-similar solution of (1) is a flow in which the shapes of the curves change homothetically and continuously to a point in finite time. A curve in the flow is called a contracting self-similar curve. Obviously, the circle is a contracting self-similar curve. It turns out that other such curves must have self-intersections. In fact, all closed contracting self-similar solutions had been completely classified by Abresch and Langer, [AL]. For our future discussion in this article, it is convenient to express their result in terms of the support function of the curve, since all these solutions are locally convex.

For each locally convex curve, $\gamma(p)$, the support function is a function

$$h(\theta) = \langle \gamma(p), \boldsymbol{\nu}(p) \rangle,$$

where θ is the angle of the outward normal and $\boldsymbol{\nu} = (\cos\theta, \sin\theta)$. The position of the curve γ in terms of h is given by

$$\gamma(p) = (h(\theta)\cos\theta - h_\theta(\theta)\sin\theta, h(\theta)\sin\theta + h_\theta(\theta)\cos\theta)\,.$$

This formulation is common in the discussions of convex plane curves. For reference, one may see [CZ]. Using the formula

$$\kappa(p) = \frac{1}{h(\theta) + h_{\theta\theta}(\theta)},$$

equation (1) can be rephrased as

$$h_t(\theta,t) = \frac{-1}{h(\theta,t) + h_{\theta\theta}(\theta,t)}$$

and the support function $h(\theta)$ of a contracting self-similar solution satisfies

$$h_{\theta\theta} + h = \frac{C}{h}, \qquad h > 0, \quad C > 0, \tag{2}$$

where C is a positive constant. Without loss of generality, from now on, we take $C = 1$. Moreover, equation (2) has a finite integral

$$h_\theta^2 + h^2 = 2\log h + C_0,$$

It is easy to see that its solutions are positive and periodic. Let us denote by $h(\theta;\alpha)$ the solution to (2) satisfying the initial conditions

$$h(0;\alpha) = \alpha > 1 \qquad \text{and} \qquad h_\theta(0;\alpha) = 0.$$

Then the main result in [AL] can be stated as follows:

Abresch-Langer Theorem. *First, the circle is the only embedded contracting self-similar curve. Second, as α increases from 1 to ∞, the period of $h(\,\cdot\,;\alpha)$ decreases strictly from $2\pi/\sqrt{2}$ to π.*

Whenever the curve determined by the support function $h(\,\cdot\,;\alpha)$ is closed, its period must be a multiple of 2π. In other words, for any pair of relatively prime positive integers m, n satisfying $\frac{1}{2} < \frac{m}{n} < \frac{1}{\sqrt{2}}$, there corresponds a unique contracting self-similar curve with n leaves in m rotations. In this article, we follow the notation of Abresch-Langer to denote such a homothetic curve by $\gamma_{m,n}$ and, likewise, its support function by $h_{m,n}$.

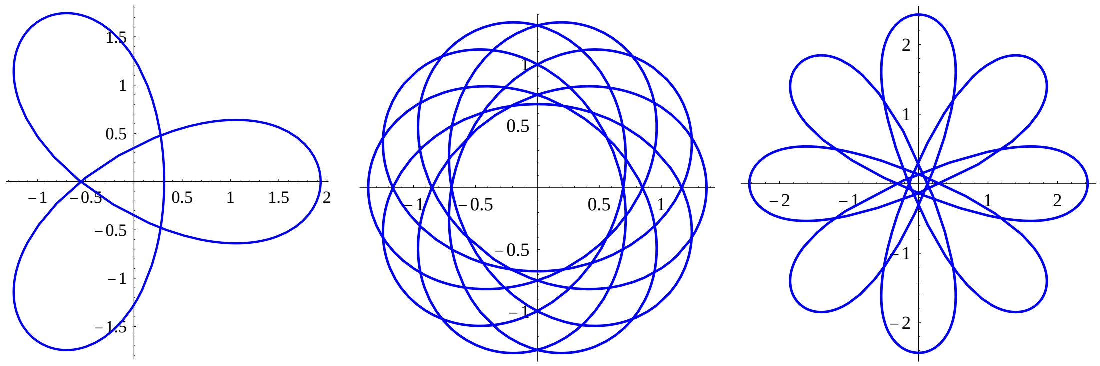


Figure 1: pictures of $\gamma_{2,3}$, $\gamma_{7,10}$, and $\gamma_{5,8}$.

Geometrically, the value α is the distance from the origin to the tip of a leave; and analytically, it is the maximum of h.

The linear stability properties of these contracting self-similar solutions are studied in [AL] and Epstein-Weinstein, [EW]. As related to the nonlinear stability of these curves, Abresch and Langer explain that they serve in a certain sense as "saddle points" between nonsingular and singular curves. More precisely, their conjecture can be stated as follows.

Conjecture. *Consider the equation* (1) *with initial data* $\gamma_0 = \gamma_{m,n} + \varepsilon \boldsymbol{\nu}$, *with* $|\varepsilon|$ *small.*

(a) *When* $\varepsilon > 0$, *the trajectory through* γ_0 *is asymptotic to an* m*-fold circle; and*

(b) *when* $\varepsilon < 0$, *the trajectory through* γ_0 *is asymptotic to a singular curve* $\Gamma_{m,n}$ *with* n *cusp points.*

They also expect the evolution process after rescaling behaves as shown in the following pictures.

Figure 2a: rescaled evolution for $\varepsilon > 0$.

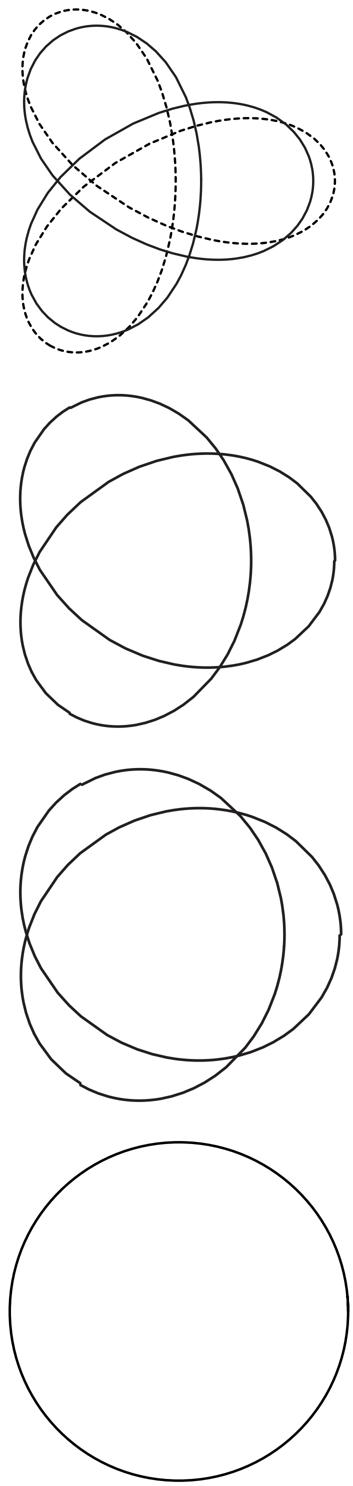
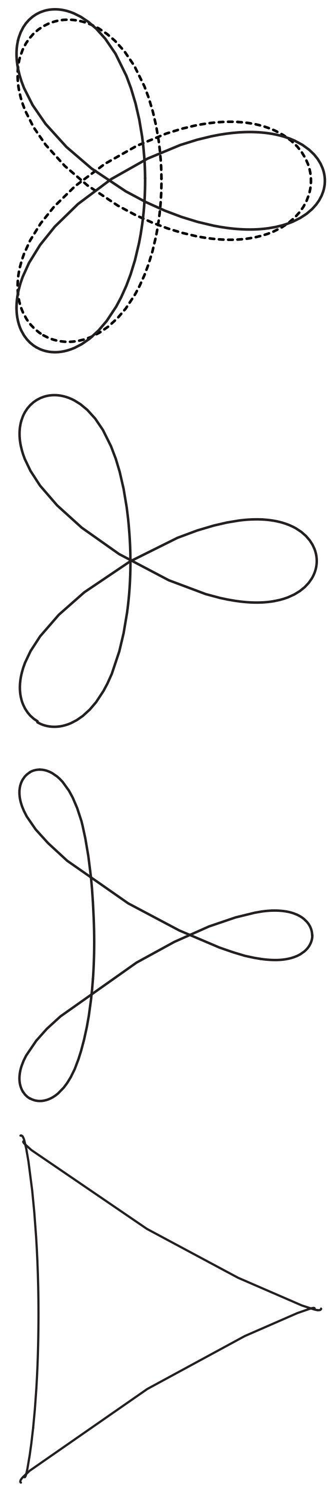


Figure 2b: rescaled evolution for $\varepsilon < 0$.



These pictures are numerically produced with a Mathematica algorithm which is mostly according to the flow with minor modification to overcome the slow convergence of the program. All the curves have been rescaled. The leftmost picture shows homothetic curve $\gamma_{2,3}$ in dash and its perturbation $\gamma_{2,3} + \varepsilon \boldsymbol{\nu}$ in solid.

We shall prove the conjecture in this article. Furthermore, we also show analytically how the flow develops singularities in case (b) and the shape and structure of the curve is the same as the qualitative description in [AL].

Main Theorem. *The Abresch-Langer conjecture is true.*

(a) *In the case that* $\varepsilon > 0$*, solution to* (1) *exists for all time* $t \in [0, \infty)$ *and it tends to an* m*-circle as* $t \to \infty$*.*

(b) *In the case that* $\varepsilon < 0$*,* n *singularity points are formed when the area of a leaf becomes zero and the curvature blows up.*

Our discussion is organized in the following way. In §1, we formulate the problem in a suitable and convenient way. Several equivalent versions of the problem will be present; each of which will be used in later discussion. The formulation of the problem in terms of support function has been given in [CZ]. In fact, such formulation provides a clean and elegant view of the topic, which is essential to our solution of the problem. In §2, we first lay the foundation for applying maximum principle to the solution. It follows naturally with the derivative estimates and the convergence of the flow to self-similar solution in the case that $\varepsilon > 0$. Then the instability of Abresch-Langer curves is shown. In §3, we deal with the case that $\varepsilon < 0$. More specifically, we first show that the solution always exists before the second stage in Figure 2b. It further exists as long as the leaves do not vanish. Since it is known that these leaves eventually eventually shrink to a point, at this moment, the singularities occur.

The discussions from our colleagues have been very encouraging and insightful in our study. We would like to specially mention K. S. Chou and Tom Wan.

1. Formulation and Normalization

First of all, let us reformulate the problem in terms of support functions of the curves. We would like to study the initial value problem of solving for $h(\theta, t)$,

$$\begin{cases} h_t(\theta, t) = \dfrac{-1}{h(\theta, t) + h_{\theta\theta}(\theta, t)}, \\ h(\theta, 0) = h_{m,n}(\theta) + \varepsilon. \end{cases}$$

In order to compare the problem with the situation at $\varepsilon = 0$, we replace $h_{m,n} + \varepsilon$ with

$$h_\varepsilon(\theta) = \left(1 + \frac{L}{m\pi}\varepsilon + \varepsilon^2\right)^{-1/2} \left(h_{m,n}(\theta) + \varepsilon\right),$$

where L is the arc length of $\gamma_{m,n}$. Next, the algebraic area of the curve γ can be determined by the support function h, namely,

$$A(h) = \frac{1}{2}\int_0^{2m\pi} h(h + h_{\theta\theta})\, \mathrm{d}\theta.$$

Then, $A(h_{m,n}) = A(h_\varepsilon) = m\pi$ with the above choice of initial support function. Let $h(\theta, t)$ be the periodic solution to the evolution equation for $\theta \in [0, 2m\pi]$ and $t \geq 0$,

$$\begin{cases} h_t(\theta, t) = \dfrac{-1}{h(\theta, t) + h_{\theta\theta}(\theta, t)}, \\ h(\theta, 0) = h_\varepsilon(\theta). \end{cases} \tag{3}$$

We then have

$$\begin{aligned} \frac{\partial A(h(\,\cdot\,, t))}{\partial t} &= \frac{1}{2}\int_0^{2m\pi} h_t(h + h_{\theta\theta}) + h(h_t + h_{t\theta\theta})\, \mathrm{d}\theta \\ &= \int_0^{2m\pi} h_t(h + h_{\theta\theta}) \qquad \text{after twice integrating by parts,} \\ &= \int_0^{2m\pi} (-1) = -2m\pi. \end{aligned}$$

In other words, $A(h) = A_0 - 2m\pi t = m\pi(1-2t)$. We further normalize the flow by letting

$$\tilde{h}(\theta, t) = \frac{1}{\sqrt{1-2t}} h(\theta, t)$$

and changing of variables from t to τ such that $\dfrac{\mathrm{d}\tau}{\mathrm{d}t} = 1/(1-2t)$. We have the following initial value problem for $\theta \in [0, 2m\pi]$ and $\tau \geq 0$,

$$\begin{cases} \tilde{h}_\tau(\theta,\tau) = -\tilde{\kappa}(\theta,\tau) + \tilde{h}(\theta,\tau), \\ \tilde{h}(\theta, 0) = h_\varepsilon(\theta). \end{cases} \tag{4}$$

As long as $A(h) > 0$, (4) is equivalent to (3). This is true in particular if $h > 0$. Under this formulation, $h_{m,n}$ is a stationary solution to (4).

Let κ and $\tilde{\kappa}$ denote the curvatures of the curves with support functions h and $\tilde{h}$ respectively. Then they satisfy the following equations, which correspond to equations (3) and (4) respectively,

$$\kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa), \tag{5}$$

$$\tilde{\kappa}_\tau = \tilde{\kappa}^2\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) - \tilde{\kappa}. \tag{6}$$

The details of the discussion in terms of support function can be referred to [CZ].

2. Outward Perturbation

In this section, we will deal with the case that $\varepsilon > 0$. This corresponds to that the initial curve is a small outward perturbation of an Abresch-Langer self-similar curve.

Lemma 1. *Let $\varepsilon > 0$ and $h(\theta, t)$ be a $\frac{2m\pi}{n}$-periodic solution to initial value problem (3). Then*

$$\max_\theta h(\theta, t) = h(0, t); \qquad \min_\theta h(\theta, t) = h(m\pi/n, t);$$

and, h is decreasing for $\theta \in [0, m\pi/n]$. Furthermore, let κ be the curvature of the curve supported by h, then $\max_\theta \kappa(\theta, t) = \kappa(0, t)$ and $\min_\theta \kappa(\theta, t) = \kappa(m\pi/n, t)$.

Proof. Firstly, for all $k \geq 1$, $\partial_\theta^k(h_\varepsilon) = \left(1 + \frac{L}{m\pi}\varepsilon + \varepsilon^2\right)^{-1/2} \partial_\theta^k(h_{m,n})$. Therefore, h_ε and $h_{m,n}$ have the same critical points with the same the extremal properties. As a result,

$$\max_\theta h_\varepsilon(\theta) = h_\varepsilon(0), \qquad \min_\theta h_\varepsilon(\theta) = h_\varepsilon(m\pi/n)$$

and h_ε decreases for $\theta \in [0, m\pi/n]$. Secondly, from the evolution equation,

$$h_{t\theta} = \frac{1}{(h + h_{\theta\theta})^2}(h_\theta + h_{\theta\theta\theta}).$$

This shows that h_θ satisfies a parabolic equation, $(h_\theta)_t = \kappa^2(h_\theta)_{\theta\theta} + \kappa^2(h_\theta)$. By Sturm oscillation theorem, [An2], the number of zeros of h_θ is non-increasing in t. Thus, for all $t > 0$, h has at most 2 critical points in $[0, m\pi/n]$.

On the other hand, by symmetry of h along $\theta = 0$ and $\theta = m\pi/n$, we must have $h_\theta = 0$ at the symmetry. Hence for each t, $h(\theta, t)$ has exactly one maximum and one minimum for $\theta \in [0, m\pi/n]$. The desired results for h follow easily.

The proof for κ is similar by simply observing that

$$\kappa_\varepsilon = \left(1 + \frac{L}{m\pi}\varepsilon + \varepsilon^2\right)^{1/2} \kappa_{m,n};$$

and the equation for κ_θ is $(\kappa_\theta)_t = \kappa^2(\kappa_\theta)_{\theta\theta} + 2\kappa\kappa_\theta(\kappa_\theta)_\theta + 3\kappa^2\kappa_\theta$. $\square$

Remark. From the above proof, we indeed have some information about the shape of each leaf of the immersed curve defined by such $h(\theta, t)$. In particular, both the longest distance (the tip of a leaf) and the shortest distance from the origin to the curve are attained along the same directions as those of $\gamma_{m,n}$.

Lemma 2. *For sufficiently small $\varepsilon > 0$, $h_\varepsilon(0) < h_{m,n}(0)$ and $h_\varepsilon(m\pi/n) > h_{m,n}(m\pi/n)$. Moreover, $\kappa_\varepsilon(0) < \kappa_{m,n}(0)$ and $\kappa_\varepsilon(m\pi/n) > \kappa_{m,n}(m\pi/n)$. All these inequalities reverse for $\varepsilon < 0$.*

Proof. By definition of h_ε and its expansion in ε, we have

$$h_\varepsilon = h_{m,n} + \left(1 - \frac{L}{2m\pi} h_{m,n}\right) \varepsilon + O(\varepsilon^2).$$

Note that $h_{m,n} = \kappa_{m,n}$, so

$$\int_0^{2m\pi} \frac{1}{h_{m,n}} = \int_0^{2m\pi} \frac{1}{\kappa_{m,n}} = \int_0^{2m\pi} [h_{m,n} + (h_{m,n})_{\theta\theta}] = \int_0^{2m\pi} h_{m,n} = L.$$

Applying the Mean Value Theorem, there is a θ_* such that $\dfrac{1}{h_{m,n}(\theta_*)}(2m\pi) = L$. According to the preceding proposition, $h_{m,n}(0) > h_{m,n}(\theta_*) > h_{m,n}(m\pi/n)$. As a consequence,

$$1 - \frac{L}{2m\pi} h_{m,n}(0) < 0, \qquad 1 - \frac{L}{2m\pi} h_{m,n}(m\pi/n) > 0.$$

The inequalities can be obtained by putting the above into the expansion of h_ε.

The proof for the inequalities of κ is similar by observing that

$$\kappa_\varepsilon = \kappa_{m,n} + \varepsilon \kappa_{m,n} \left(\frac{L}{2m\pi} - \kappa_{m,n}\right) + \mathrm{O}(\varepsilon^2),$$

and applying the Mean Value Theorem to $\displaystyle\int_0^{2m\pi} \kappa_{m,n} = \int_0^{2m\pi} h_{m,n} = L.$ ∎

Thus in the case that $\varepsilon > 0$, it follows from the maximum principle, applied to (4) and (6), that there are positive uniform upper and lower bounds for $\tilde{h}$ and $\tilde{\kappa}$ for all $\tau \geq 0$. As a consequence of parabolic regularity theory, their higher derivatives are also uniformly bounded. From these estimates, we infer the long time existence of (4) and (6).

Proposition 3. *As $\tau \to \infty$, the solution $\tilde{h}$ to equation* (4) *with $\varepsilon > 0$ approaches a stationary solution, which is either $h_{m,n}$ itself or the m-circle.*

Proof. We consider the entropy $\mathcal{E} = \int_0^{2m\pi} \log \tilde{\kappa}$ for the normalized flow (6). Then

$$\begin{aligned}
\mathcal{E}'(\tau) &= \int_0^{2m\pi} \frac{\tilde{\kappa}_\tau}{\tilde{\kappa}} = \int_0^{2m\pi} \left[\tilde{\kappa}\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) - 1\right] \\
&= -2m\pi + u(\tau),
\end{aligned}$$

where $u(\tau) = \int_0^{2m\pi} \tilde{\kappa}\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right)$.

$$\begin{aligned}
u'(\tau) &= \int_0^{2m\pi} \tilde{\kappa}_\tau\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) + \tilde{\kappa}\left(\tilde{\kappa}_\tau + \tilde{\kappa}_{\theta\theta\tau}\right) \\
&= 2\int_0^{2m\pi} \tilde{\kappa}_\tau\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) \qquad\qquad \text{using integration by parts} \\
&= 2\int_0^{2m\pi} \left[\tilde{\kappa}^2\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) - \tilde{\kappa}\right]\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) \\
&= 2\int_0^{2m\pi} \tilde{\kappa}^2\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right)^2 - 2\int_0^{2m\pi} \tilde{\kappa}\left(\tilde{\kappa} + \tilde{\kappa}_{\theta\theta}\right) \\
&\geq \frac{2}{m\pi} u(\tau)\left(u(\tau) - 2m\pi\right).
\end{aligned}$$

Suppose there is a τ_1 such that $u(\tau_1) \geq 2m\pi$. Then $u(\tau)$ blows up in finite time and so does $\mathcal{E}(\tau)$. This contradicts the boundedness of $\mathcal{E}$. Thus, $u(\tau) \leq 2m\pi$ and $\mathcal{E}$ is decreasing in τ as in the embedded case, [GH].

Using this property, we conclude that there is a sequence, after passing to a subsequence, $\tau_j \to \infty$ such that $\mathcal{E}'(\tau_j) \to 0$. Thus, using the uniform bounds on $\tilde{\kappa}$, we see that $\tilde{\kappa}_\tau \to 0$ as $\tau \to \infty$ which in turns implies $\tilde{\kappa}$ and hence $\tilde{h}$ converge. $\square$

We will rule out that it converges back to an Abresch-Langer curve. To prepare for this, we first write down some expansions. Let κ_ε be the curvature corresponding to h_ε. We have

$$h_\varepsilon = h_{m,n} + \varepsilon\left(1 - \frac{L}{2m\pi}h_{m,n}\right) + \varepsilon^2\left(\frac{-1}{2}h_{m,n} - \frac{L}{2m\pi} + \frac{3L^2}{8m^2\pi^2}h_{m,n}\right) + \mathrm{O}(\varepsilon^3);$$
$$\kappa_\varepsilon = \kappa_{m,n}\left[1 + \varepsilon\left(\frac{L}{2m\pi} - \kappa_{m,n}\right) + \varepsilon^2\left(\frac{1}{2} + \kappa_{m,n}{}^2 - \frac{L^2}{8m^2\pi^2} - \frac{L}{2m\pi}\kappa_{m,n}\right)\right] + \mathrm{O}(\varepsilon^3).$$

Besides the entropy $\mathcal{E}$ we have seen before, there is another useful functional which is also decreasing along the flow. Let

$$\mathcal{E}(\varepsilon) = \int_0^{2m\pi} \log \kappa_\varepsilon, \qquad\qquad \mathcal{F}(\varepsilon) = \int_0^{2m\pi} \log h_\varepsilon.$$

Then

$$\mathcal{E}'(\varepsilon) = \int_0^{2m\pi} \frac{\kappa_{m,n}}{\kappa_\varepsilon}\left[\frac{L}{2m\pi} - \kappa_{m,n} + \varepsilon\left(1 + 2\kappa_{m,n}{}^2 - \frac{L^2}{4m^2\pi^2} - \frac{L\kappa_{m,n}}{m\pi}\right) + \mathrm{O}(\varepsilon^2)\right];$$
$$\mathcal{E}''(\varepsilon) = \int_0^{2m\pi} \frac{\kappa_{m,n}}{\kappa_\varepsilon}\left(1 + 2\kappa_{m,n}{}^2 - \frac{L^2}{4m^2\pi^2} - \frac{L\kappa_{m,n}}{m\pi}\right) - \frac{\kappa_{m,n}{}^2}{\kappa_\varepsilon{}^2}\left(\frac{L}{2m\pi} - \kappa_{m,n}\right)^2 + \mathrm{O}(\varepsilon).$$

It follows that, using $\kappa_{m,n} = h_{m,n}$,

$$\begin{aligned}\mathcal{E}''(\varepsilon) &= 2m\pi - \frac{L^2}{m\pi} + \int_0^{2m\pi} \kappa_{m,n}{}^2 \\ &= 2m\pi\left[1 - 2\left(⨍ h_{m,n}\right)^2 + ⨍ h_{m,n}{}^2\right] + \mathrm{O}(\varepsilon).\end{aligned}$$

For $\mathcal{F}$, we may also work out the same calculations.

$$\mathcal{F}'(\varepsilon) = \int_0^{2m\pi} \frac{1}{h_{m,n}}\left[1 - \frac{L}{2m\pi}h_{m,n} + \varepsilon\left(-h_{m,n} - \frac{L}{m\pi} + \frac{3L^2}{4m^2\pi^2}h_{m,n}\right) + \mathrm{O}(\varepsilon^2)\right];$$

$$\mathcal{F}''(\varepsilon) = \int_0^{2m\pi} \frac{1}{h_{m,n}}\left(-h_{m,n} - \frac{L}{m\pi} + \frac{3L^2}{4m^2\pi^2}h_{m,n}\right) - \frac{1}{{h_{m,n}}^2}\left(1 - \frac{L}{2m\pi}h_{m,n}\right)^2 + \mathrm{O}(\varepsilon).$$

Using the fact that $1/h_{m,n} = 1/\kappa_{m,n} = h_{m,n} + (h_{m,n})_{\theta\theta}$, we have

$$\mathcal{F}''(\varepsilon) = -2m\pi\left[1 - \left(\fint \frac{1}{h_{m,n}}\right)^2 + \fint \frac{1}{{h_{m,n}}^2}\right] + \mathrm{O}(\varepsilon).$$

With these calculations, we have the following nonlinear instability result.

Proposition 4. $(\mathcal{E} + \mathcal{F})(\varepsilon) < 0$ *for all sufficiently small* $\varepsilon > 0$.

Proof. We consider the functional $\mathcal{E} + \mathcal{F}$ and already have

$$(\mathcal{E} + \mathcal{F})''(\varepsilon) = 2m\pi\left[\fint {h_{m,n}}^2 - \fint \frac{1}{{h_{m,n}}^2}\right] + \mathrm{O}(\varepsilon).$$

Here,

$$\begin{aligned}
\fint \frac{1}{{h_{m,n}}^2} &= \fint [h_{m,n} + (h_{m,n})_{\theta\theta}]^2 \\
&= \fint {h_{m,n}}^2 + 2\fint h_{m,n}(h_{m,n})_{\theta\theta} + \fint (h_{m,n})^2_{\theta\theta} \\
&= \fint {h_{m,n}}^2 - 2\fint (h_{m,n})^2_{\theta} + \fint (h_{m,n})^2_{\theta\theta}.
\end{aligned}$$

By Poincaré Inequality, we have $\fint (h_{m,n})^2_{\theta\theta} \geq \left(\frac{2\pi}{2m\pi/n}\right)^2 \fint (h_{m,n})^2_{\theta}$. Combining the above, together with $\frac{m}{n} < \frac{1}{\sqrt{2}}$, we have

$$(\mathcal{E} + \mathcal{F})''(\varepsilon) = \left[2 - \frac{n^2}{m^2}\right] \fint (h_{m,n})^2_{\theta} + \mathrm{O}(\varepsilon) < 0,$$

for sufficiently small ε. $\square$

Now, we can finish the proof of part (a) in the main theorem. We have already known that the flow $\tilde{\gamma}$ exists for all $\tau \geq 0$. By proposition 3, it converges either to

the m-circle or back to its initial curve. As $\mathcal{E} + \mathcal{F}$ is non-increasing along the flow, it follows from proposition 4 that the latter is impossible. So, it must converge to the m-circle and (a) holds. Note that we work on equation (4) which is obtained from normalization by the algebraic area. It is equivalent to that by arc length.

3. Inward Perturbation

In this section, we will show that, for $\varepsilon < 0$, the singularities occur exactly when the area of a leaf vanishes and the curvature of the curve blows up. We first establish some analytical results. The following lemma basically lays the foundation to show that in the evolution, the leaves eventually "shrink" and exclude the origin.

Lemma 5. *Let $\tilde{h}(\theta, t)$ be a solution to the initial value problem (4). If $\tilde{h} > 0$ for all τ in an interval, then $\tilde{h}$ is also uniformly bounded above.*

Proof. Modifying an argument in [GH], we consider the following quantity, $W(\tau)$.

$$
\begin{aligned}
W(\tau) &= \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\tilde{\kappa}(\theta,\tau)} = \int_{-\pi/2}^{\pi/2} \left(\tilde{h} + \tilde{h}_{\theta\theta}\right)\cos\theta \\
&= \int_{-\pi/2}^{\pi/2} \tilde{h}\cos\theta + \tilde{h}_\theta \cos\theta\Big|_{-\pi/2}^{\pi/2} + \tilde{h}\sin\theta\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \tilde{h}\cos\theta \\
&= \tilde{h}(\pi/2) + \tilde{h}(-\pi/2).
\end{aligned}
$$

Therefore, $W(\tau)$ is the width (measured perpendicular to the longest axis) of a leaf of the curve defined by h. By Jensen's Inequality,

$$\begin{aligned}\log W(\tau) &\geq \int_{-\pi/2}^{\pi/2} \log\left(\frac{\cos\theta}{\tilde{\kappa}(\theta,\tau)}\right) \\ &= \int_{-\pi/2}^{\pi/2} \log\cos\theta - \int_{-\pi/2}^{\pi/2} \log\tilde{\kappa} \\ &= \log C - \mathcal{E}(\tau) \geq \log C - \mathcal{E}(0),\end{aligned}$$

for some constant $C > 0$, because $\mathcal{E}$ is decreasing (proved in proposition 3). Thus, $W(\tau)$ is uniformly bounded below. Since

$$W(\tau)\cdot\max_\theta \tilde{h}(\theta,\tau) \leq \text{the area of a leaf} \leq A(h) = m\pi,$$

it follows that $\max_\theta \tilde{h}(\theta,\tau)$ has a uniform upper bound. □

Now, we may use this lemma to show that for $\varepsilon < 0$, at some time, a curve in the flow will pass through the origin.

Proposition 6. *Let $\tilde{h}(\theta,\tau)$ be the solution to the initial value problem* (4) *with $\varepsilon < 0$. Then there is a time τ_1 such that $\min_\theta \tilde{h}(\theta,\tau_1) = 0$ and $\tilde{h} > 0$ for $\tau \in [0,\tau_1)$.*

Proof. Suppose on the contrary that $\tilde{h} > 0$ for all $\tau \geq 0$. Locally express the tip of a leaf as a concave graph. Specifically, we write

$$\tilde{\gamma}(\theta(x),\tau) = (x, u(x,\tau)), \qquad x \in (-\ell,\ell),$$

such that $\theta(0) = 0$. By assumption and the preceding lemma, there is M such that $\max_x u \leq M$ for all τ. Note that the equation on $\tilde{\gamma}$ corresponding to equation (4) is

$$\tilde{\gamma}_\tau = \tilde{\kappa}\boldsymbol{\nu} + \tilde{\gamma}.$$

Under this local parametrization, we have

$$\begin{aligned}\tilde{\gamma}_\tau &= \frac{\partial x}{\partial \tau}(1, u_x) + (0, u_\tau);\\ \boldsymbol{\nu} &= \frac{1}{\sqrt{1+u_x^2}}(-u_x, 1);\\ \tilde{\kappa} &= \frac{u_{xx}}{(1+u_x^2)^{3/2}}.\end{aligned}$$

Therefore, we have the parabolic equation

$$u_\tau = \frac{u_{xx}}{1+u_x^2} - xu_x + u. \tag{7}$$

Fixed a small $\xi > 0$, we consider the sub-interval $(-\ell + \xi, \ell - \xi)$. By concavity of the graph, we have

$$\sup \{|u_x| : x \in (-\ell+\xi, \ell-\xi)\} \leq \frac{1}{\xi} \operatorname{osc}_{(-\ell,\ell)} u \leq \frac{M}{\xi}$$

Hence, (7) is uniformly parabolic. By parabolic regularity theory, we know that all higher derivatives of u are uniformly bounded in $(-\ell + \xi, \ell - \xi)$. In particular, the curvature at the tip, when $x = 0$, is uniformly bounded. By propositions 3 and 4, we know that $\tilde{\gamma}$ converges to the m-circle, contradicting lemma 2 and the maximum principle. □

The contradiction shows that $\min_\theta \tilde{h}(\theta, \tau_1) = 0$. Clearly, shortly after τ_1, $\min_\theta \tilde{h}(\theta, \tau) < 0$ and the evolution of the curve enters the third stage in Figure 2b.

Next, we would like to make sure the solution to problems (3) and (4) exist as long as the leaves do not shrink to points. Since the algebraic area may become negative, it is necessary to look at the unnormalized equation (3).

Proposition 7. *Let h be a solution to the initial value problem* (3)*. It exists as long as the area enclosed by a leaf does not vanish.*

Proof. Suppose otherwise, then there is $t_* > 0$ such that actual area of a leaf $\geq C > 0$ and $\lim_{t \to t_*^-} \kappa(t) = \infty$. We represent the curve at time t, $\gamma(\,\cdot\,, t)$ as a concave graph as before. There is $\delta > 0$ such that for $t \in (t_* - \delta, t_*)$ and $x \in (-\ell + \xi, \ell - \xi)$, we have the evolution equation

$$u_t = \frac{u_{xx}}{\sqrt{1 + u_x^2}},$$

which is equivalent to (3); see, for example, [CZ]. Again, by similar argument as above, u_x is uniformly bounded for $x \in (-\ell + \xi, \ell - \xi)$ and $t \in (t_* - \delta, t_*)$. Thus, we obtain a uniform bound for $\kappa(0, t) = \left. \frac{u_{xx}}{\sqrt{1 + u_x^2}} \right|_{x=0}$. By continuity, as $t \to t_*^-$, $\kappa(0, t) \to$ a finite value, which is a contradiction. □

Corollary 8. *Let t_1 be corresponding to τ_1 in proposition* 6*. There is $t_\infty > t_1$ such that every leaf of the curve shrinks to a point.*

Proof. Since $h(\theta, t_1) > 0$ for all $m\pi/n \neq \theta \in [0, 2m\pi/n]$, we have $A(h(\,\cdot\,, t_1)) > 0$. By continuity, there is $\delta > 0$ such that $A(h(\,\cdot\,, t)) > 0$ for $t < t_1 + \delta$. Therefore, the area of a leaf of the curve at time $t < t_1 + \delta$ is still positve. According to the above proposition, solution to (3) still exists for $t < t_1 + \delta$. On the other hand, it is easy to see that the rate of change of the area of a leaf is less than $-\pi$, [AL]. Thus, there is a time $t_\infty > t_1$ that all the m leaves disappear. □

Remark. Recently, an affine version of the curve shortening flow is studied by B. Andrews, [Ad], also de Lime and Montenegro, [LM]. In the latter paper, a classification theorem parallel to the Abresch-Langer theorem for contracting self-similar curve along the affine flow is established. It is interesting to study whether the saddle point property still holds for these curves.

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